1) By Euler's Formula, where v is a fixed direction and θ is the angle wrt. v.

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Then $\displaystyle\int_0^{\pi} k_n(\theta)\,d\theta = \int_0^{\pi} (k_1\cos^2\theta + k_2\sin^2\theta)\,d\theta$

$$= \int_0^{\pi} k_1\left(\frac{1+\cos 2\theta}{2}\right) + k_2\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{k_1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi} + \frac{k_2}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi}$$

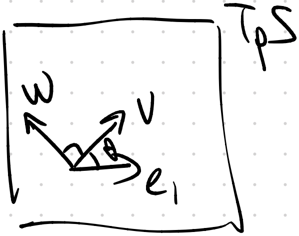
$$= \frac{k_1\pi}{2} + \frac{k_2\pi}{2} = \frac{\pi}{2}(k_1+k_2)$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi} k_n(\theta)\,d\theta = \frac{1}{2}(k_1+k_2) = H.$$

//.

2) let $p \in S$, $v, w \in T_pS$. s.t. $v \perp w$. ie. $\langle v, w \rangle = 0$.

let θ be the angle between v and some chosen e_1



Then since $v \perp w$, we have that angle between w and e_1 is given by $\theta + \frac{\pi}{2}$.

Then $k_n(v) + k_n(w)$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta + k_1 \cos^2\left(\theta + \tfrac{\pi}{2}\right) + k_2 \sin^2\left(\theta + \tfrac{\pi}{2}\right)$$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta + k_1\left(\cos\theta \overset{0}{\cancel{\cos\tfrac{\pi}{2}}} - \sin\theta \overset{1}{\cancel{\sin\tfrac{\pi}{2}}}\right)^2 + k_2\left(\sin\theta \overset{0}{\cancel{\cos\tfrac{\pi}{2}}} + \cos\theta \overset{1}{\cancel{\sin\tfrac{\pi}{2}}}\right)^2$$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta + k_1 \sin^2\theta + k_2 \cos^2\theta$$

$$= k_1 + k_2$$ which is constant //.

3) $X(x,y)=(x,y,axy)$.

$X_x=(1,0,ay)$ $X_y=(0,1,ax)$

$X_{xx}=(0,0,0)$, $X_{xy}=X_{yx}=(0,0,a)$ $X_{yy}=(0,0,0)$

$E=\langle X_x,X_x\rangle=1+a^2y^2$ $F=\langle X_x,X_y\rangle=a^2xy$.

$G=\langle X_y,X_y\rangle=1+a^2x^2$.

$$EG-F^2=(1+a^2y^2)(1+a^2x^2)-a^4x^2y^2$$

$$=1+a^2(x^2+y^2)+a^4x^2y^2-a^4x^2y^2=1+a^2(x^2+y^2).$$

$X_x\times X_y=(-ay,-ax,1)$

$$N=\frac{1}{\sqrt{1+a^2(x^2+y^2)}}(-ay,-ax,1)$$

$e=\langle N,X_{xx}\rangle=0$, $g=\langle N,X_{yy}\rangle=0$

$$f=\langle N,X_{xy}\rangle=\frac{a}{\sqrt{1+a^2(x^2+y^2)}}$$

$$K=\frac{eg-f^2}{EG-F^2}=\boxed{\frac{-a^2}{(1+a^2(x^2+y^2))^2}}$$

$$H=\frac{1}{2}\frac{\overset{0}{\cancel{e}}G-2fF+\cancel{g}\overset{0}{E}}{EG-F^2}=\frac{-fF}{EG-F^2}=\boxed{\frac{-a^3xy}{(1+a^2(x^2+y^2))^{3/2}}}$$

4) i) $d(fN)(v) = \frac{d}{dt}\left(f(\alpha(t))N(\alpha(t))\right)\Big|_{t=0}$

$= \frac{d}{dt} f(\alpha(t))\Big|_{t=0} N(\alpha(0)) + f(\alpha(0)) \frac{d}{dt} N(\alpha(t))\Big|_{t=0}$

$= f'(\alpha(t))\alpha'(t)\Big|_{t=0} N(p) + f(p)\, dN_p(v)$

$= \langle df(p), v \rangle N(p) + f(p)\, dN_p(v)$

Then $\langle d(fN)(v_1) \times d(fN)(v_2), N \rangle$

$= \det\left(d(fN)(v_1), d(fN)(v_2), N\right)$

$= \det\big(\langle df(p), v_1\rangle N(p) + f(p)\, dN_p(v_1),$
$\langle df(p), v_2\rangle N(p) + f(p)\, dN_p(v_2), N\big)$

$= \det\left(f(p)\, dN_p(v_1), f(p)\, dN_p(v_2), N\right)$

$= f^2 \langle dN_p(v_1) \times dN_p(v_2), N \rangle$

let $S_p = [h_{ij}]$ be the matrix of the shape operator at p.

Recall $S_p = -dN_p$, so $dN_p = [-h_{ij}]$ and in the basis $\{v_1, v_2\}$

$dN_p(v_1) = -h_{11}v_1 - h_{21}v_2$, $dN_p(v_2) = -h_{12}v_1 - h_{22}v_2$.

So $dN_p(v_1) \times dN_p(v_2) = (-h_{11}v_1 - h_{21}v_2) \times (-h_{12}v_1 - h_{22}v_2)$

$$= (\cancel{-h_{11}v_1 \times -h_{12}v_1})^{0} + (-h_{11}v_1 \times -h_{22}v_2) + (-h_{21}v_2 \times -h_{12}v_1)$$

$$+ (\cancel{-h_{21}v_2 \times -h_{22}v_2})^{0}$$

$$= (h_{11}h_{22} - h_{12}h_{21})\, v_1 \times v_2$$ ← since $\{v_1, v_2\}$ is an onbs, $N = v_1 \times v_2$.

$$= \det(S_p) N$$

$$= K(p) N.$$

$$\Rightarrow \langle dN_p(v_1) \times dN_p(v_2), N \rangle = K(p)$$

$$\Rightarrow \frac{\langle d(fN)(v_1) \times d(fN)(v_2), N \rangle}{f^2} = \frac{f^2 \langle dN_p(v_1) \times dN_p(v_2), N \rangle}{f^2}$$

$$= K(p) \text{ as required}$$

vi) $h(x,y,z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$f(x,y,z) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{1/2} \Bigg|_h.$$

we can take

$$N = \frac{dh}{|dh|} = \frac{dh}{\left(\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2\right)^{1/2}} = \frac{dh}{\left(4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^{1/2}} = \frac{dh}{2f}$$

For a vector $v=(v_1,v_2,v_3)$, we have

$$d(fN)(v) = \frac{1}{2}dh(v) = \frac{1}{2}\left(\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz\right)(v_1,v_2,v_3)$$
$$= \left(\frac{v_1}{a^2},\frac{v_2}{b^2},\frac{v_3}{c^2}\right).$$

let $\{u,v\}$ be an onbs, $N=u\times v=(n_1,n_2,n_3)$, then by (i), we have

$$K=\frac{1}{f^2}\begin{vmatrix} \frac{u_1}{a^2} & \frac{u_2}{b^2} & \frac{u_3}{c^2} \\ \frac{v_1}{a^2} & \frac{v_2}{b^2} & \frac{v_3}{c^2} \\ n_1 & n_2 & n_3 \end{vmatrix} = \frac{1}{f^2a^2b^2c^2}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ a^2n_1 & b^2n_2 & c^2n_3 \end{vmatrix}$$

$$= \frac{1}{f^2a^2b^2c^2}\langle u\times v, (a^2n_1, b^2n_2, c^2n_3)\rangle$$

$$= \frac{1}{f^2a^2b^2c^2}\langle N, (a^2n_1, b^2n_2, c^2n_3)\rangle$$

$$= \frac{1}{f^2a^2b^2c^2}(a^2n_1^2+b^2n_2^2+c^2n_3^2)$$

$$= \frac{1}{f^2a^2b^2c^2}\langle (a^2,b^2,c^2), (n_1^2,n_2^2,n_3^2)\rangle.$$

From $N=\frac{dh}{2f}$, we get

$$(n_1,n_2,n_3) = \left(\frac{x}{fa^2},\frac{y}{fb^2},\frac{z}{fc^2}\right)$$

$$\Rightarrow (n_1^2, n_2^2, n_3^2) = \left(\frac{x^2}{f^2a^4}, \frac{y^2}{f^2b^4}, \frac{z^2}{f^2c^4}\right)$$

So $\langle (a^2, b^2, c^2), (n_1^2, n_2^2, n_3^2) \rangle$

$$= \left\langle (a^2, b^2, c^2), \left(\frac{x^2}{f^2a^4}, \frac{y^2}{f^2b^4}, \frac{z^2}{f^2c^4}\right) \right\rangle$$

$$= \frac{x^2}{f^2a^2} + \frac{y^2}{f^2b^2} + \frac{z^2}{f^2c^2}$$

So $K = \frac{1}{f^2a^2b^2c^2}\left(\frac{x^2}{f^2a^2} + \frac{y^2}{f^2b^2} + \frac{z^2}{f^2c^2}\right) = \frac{1}{f^4a^2b^2c^2}\underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)}_{=h=1}$

$$= \frac{1}{f^4a^2b^2c^2} \text{ as required } //.$$